$$\pi r = \pi a\left(1 - \frac{e^2}{2^2} - \frac{1^2.3.e^4}{2^2.4^2} - \frac{1^2.3^2.5.e^6}{2^2.4^2.6^2} - \right).$$

Solving by trial, $e=.895$, and $b=1.784$.

At a distance of z from the circular end of the pipe, the section is an ellipse, the semi-axes of which are found to be

$$\frac{hr-rz+az}{h} \text{ and } \frac{hr-rz+bz}{h}.$$

Hence the volume is

$$\int_0^h \pi\left(\frac{hr-rz+az}{h}\right)\left(\frac{hr-rz+bz}{h}\right)dz = \frac{\pi h}{6}(ar+br+2r^2+2ab) = 198.5\pi.$$

Hence the loss $=216\pi-198.5\pi=17.5\pi=55$ cubic inches.

MISCELLANEOUS.

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82. Proposed by A. H. BELL, Hillsboro, Ill.

Four spheres of equal radii $=r=5$, are in contact, and form a triangular pyramid. How large is the sphere that can be placed in the middle and be in contact with the four spheres.

Solution by J. W. YOUNG, Fellow in Mathematics, Cornell University, Ithaca, N. Y., and J. SCHEFFER, A. M., Hagerstown, Md.

Let A, B, C, B' (Fig. 1) be the centers of the four spheres. They evidently form the corners of a regular tetrahedron. Fig. 2 is a picture of a plane section of the pyramid of spheres, passed through the points $AB'L$, where L is the point of tangency of the two spheres (C, B).

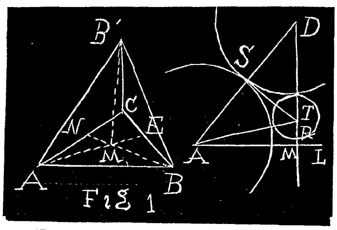

Fig. 1. Fig. 2.

From Fig. 1,

$$\left.\begin{array}{l} AN/AM=\sin 60^\circ \\ AN/AB'=\cos 60^\circ \end{array}\right\}.$$

$\therefore AB'/AM=\tan 60^\circ=\sqrt{3}=\sec DAM$.

In Fig. 2, then, $\angle DAM$ is $\sec^{-1}\sqrt{3}$. It is clear that the required small sphere must have its center on DM and must touch both spheres (A, D). Let $\angle ADM=\theta$.

Then $\sin\theta=1/\sqrt{3}$, $\cos\theta=\sqrt{\tfrac{2}{3}}$, $DT/r=\sqrt{\tfrac{3}{2}}$.

$\therefore DT=(r/2)\sqrt{6}$.

$\therefore RT=DT-r=(r/2)(\sqrt{6}-2)=$radius of small sphere.

$r=5$ gives $RT=1.1238$.

85. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Prove that at least one of the three sides of a rational right triangle must be divisible by 5.

Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, O.

In THE AMERICAN MATHEMATICAL MONTHLY, No. 1, Vol. I, Dr. Dickson gives the following formulæ for finding lowest integers representing the sides of a right triangle: $m+\sqrt{(2mn)}$, $n+\sqrt{(2mn)}$, $m+n+\sqrt{(2mn)}$, in which m and n are shown to be integers, prime to each other, the one an odd square, the other twice any square, except such as would make m and n have a common factor.

We may represent, then, m and n by $(2r+1)^2$ and $2s^2$, respectively.

Now, if any one of the three sides of a rational right triangle is divisible by 5, so then also is their product; and, conversely.

The product of the three sides, represented by the above formulae, is $5m^2n+5mn^2+5mn\sqrt{(2mn)}+(m^2+n^2(\sqrt{(2mn)}$, the first three terms of which are evidently divisible by 5. Then, substituting in the last term for m and n their equals, $(2r+1)^2$ and $2s^2$, we have

$$2s(2r+1)[(2r+1)^4+4s^4]\ldots.(1).$$

Again, all possible integers may be represented by $5k+1$, $5k+2$, $5k+3$, $5k+4$, and $5k+5$.

It is plain that if $s=5k+5$, (1) will be divisible by 5, no matter what value r may have. So, too, if $r=5k+2$, no matter what s may equal. We still have to show that the last factor of (1) is divisible by 5 when s has any other value than $5k+5$, while at the same time r has any other value than $5k+2$.

Note that under these conditions all the literal terms of $(2r+1)^4$ and $4s^4$ are divisible by (5), while the numerical term of $(2r+1)^4$ always ends with 1, and that of $4s^4$ with 4; hence the numerical term of $(2r+1)^4+4s^4$ always ends with 5. Therefore, in any case, (1) is divisible by 5, which proves the proposition.

COROLLARY. By the same method, it may be proven that one of the numbers representing the legs of a rational right triangle must be divisible by 3. Dr. Dickson has shown that one must be divisible also by 4.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

134. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

A certain piece of land is surrounded by a four-board fence, the boards being 16 feet long. The number of acres in the land equals the number of boards in the fence. How many acres in the land ?

135. Proposed by NELSON L. RORAY, Brigdeton, N. J.

If 6 is one-half of 10, what part of 20 is 12 ? Also what part of 30 is 10 ?

⁂ Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.